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On the gauge invariance of the Chern-Simons action for N D-branes

Calin Ciocarlie

California Institute of Technology
Pasadena, CA 91125, USA
E-mail: calin@theory.caltech.edu

Abstract: In this short note we provide a proof that the Chern-Simons part of the action for N D-branes is invariant under gauge transformations of the RR fields of the type $C_p \to C_p + d\Lambda_{p-1}$, and rewrite the action in a form that makes this symmetry manifest.

Keywords: D-branes, Bosonic Strings.

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Contents

1. Introduction

Using the principle of consistency under T-duality transformation, the authors of [1, 2] extended the Chern-Simons part of the action for a D_{p-1}-brane to the case of N coincident D_{p-1}-branes. The extended action contains extra terms that, in general, give a non-trivial coupling between the N D_{p-1}-branes and a higher rank RR form, C_{p+2k}. As mentioned in [3], it is not obvious whether the extended action is still invariant under gauge transformations of the type, $C_p \to C_p + d\Lambda_{p-1}$. It is the purpose of this note to investigate this question. It was not clear, a priori, whether to expect this to work. The fact that it does seems quite remarkable.

The world-volume action for the D_{p-1}-branes will be written in the static gauge: one can use spacetime diffeomorphisms to define the fiducial world-volume as $x^i = 0$, $i = p, \ldots, 9$, and world-volume diffeomorphisms to match the coordinates of the branes with the remaining space-time coordinates, i.e. $\sigma^a = x^a$, $a = 0, \ldots, p-1$. The transverse displacements of the branes are $\Delta x^i = (2\pi\alpha')\phi^i \equiv \lambda\phi^i$, where ϕ^i is an $N \times N$ matrix.

The Chern-Simons term for N coincident D_{p-1}-branes is given by [1],

$$S_{\mathrm{CS}} = \mu_{p-1} \int \mathrm{Str}\left(P\left[e^{i\lambda i_\phi i_\phi}\left(\Sigma C^{(n)} e^{B}\right)\right] e^{\lambda F}\right), \tag{1.1}$$

$P(\cdots)$ represents the pullback, $i_\phi i_\phi$ defines an inner product, e.g. $i_\phi i_\phi C^{(2)} = \frac{1}{2}[\phi^j, \phi^i]$ $C^{(2)}_{ij}$, F_{ab} is the gauge field strength living on the D-brane, and σ's are the coordinates parallel to the directions of the branes.

In this action, the background fields are considered to be functionals of the non-abelian scalars ϕ's, as suggested in [4], while the pull-backs are defined in terms of covariant derivatives, $D_a\phi^i$, as in [5]. Furthermore the action includes a symmetrized trace prescription: we have to take a symmetrized average over all orderings of ϕ^i, $D_a\phi^i$, F_{ab}, and pairs of $\underline{\phi^{2k'}\phi^{2k'-1}}$ from the inner product. This prescription is in agreement with results obtained in [6] from matrix theory considerations. For simplicity, the gauge field living on the brane (F_{ab}) and the background NS-NS field B, are initially taken to be 0. Even for this simplified case, the demonstration of gauge invariance is rather long and subtle. We have tried to make it as clear and simple as possible. Later on we will generalize the proof to non-zero F. Gauge invariant expressions will be given in eq. (2.32), for $F = 0$, and in (3.6) for $F \neq 0$. At the end, we will also explain how to include a B-field.

2. Gauge invariance, $F = 0$ case

In the following, we will show that the coupling between N D_{p-1}-branes and a C_{p+2k} RR form is invariant under the transformation $C \to C + d\Lambda$. A particular case of this problem was proved in [7], in a matrix theory context, working in the momentum basis. Here, we generalize, considering D_{p-1}-branes instead of D_0-branes, with non-trivial pull-back and F_{ab} terms. (Non-zero F will be considered in the next section.)

Specializing eq. (1.1) to the case $F = B = 0$, the coupling between a C_{p+2k} RR form and N D_{p-1}-branes is given by:

$$\mu_{p-1} \int \mathrm{Str}\left(P\left[\frac{(i\lambda i_\phi i_\phi)^k}{k!} C_{p+2k} \right]\right) . \tag{2.1}$$

Each of the RR fields C_{p+2k} are functionals of the transverse coordinates ϕ:

$$C(\sigma, \phi) = e^{\lambda \phi^i \partial_{x^i}} C^0(\sigma, x^i)|_{x^i=0} = \sum_{n,i_n} \frac{\lambda^n}{n!} \phi^{i_1} \cdots \phi^{i_n} \partial_{x^{i_1}} \cdots \partial_{x^{i_n}} C^0(\sigma, x^i)|_{x^i=0} , \tag{2.2}$$

where $C^0(\sigma, x^i)$ is the background RR field. If $\lambda\phi^i$ are the transverse displacements of the branes, the pullback of a p form, Ω_p, in the static gauge is:

$$[P(\Omega_p)]_{a_1\cdots a_p} = \Omega_{\mu_1\cdots\mu_p} \left(\delta^{\mu_1}_{a_1} I_N + \lambda \frac{\partial \phi^{\mu_1}}{\partial \sigma^{a_1}} \right) \cdots \left(\delta^{\mu_p}_{a_p} I_N + \lambda \frac{\partial \phi^{\mu_p}}{\partial \sigma^{a_p}} \right) , \tag{2.3}$$

where I_N is an $N \times N$ unit matrix, and Ω_p should be considered a functional of the ϕ's. The indices μ's run over all coordinates, so we will take $\phi^\mu = 0$ for the μ's parallel to the direction of the branes. As defined in the previous equation the pullback of an antisymmetric form is not necessarily an antisymmetric form since, as $N \times N$ matrices, $\partial_{a_i}\phi^i$ do not commute in general. However, as part of the symmetrized trace prescription we should take a symmetrized average over all orderings of $\partial_{a_i}\phi^i$, thus enforcing antisymmetry on the a's.

With antisymmetry enforced on the a's, (2.3) becomes:

$$[P(\Omega_p)]_{[a_1\cdots a_p]} = \Omega_{a_1\cdots a_p} + \lambda p \Omega_{i_1[a_2\cdots a_p}\partial_{a_1]}\phi^{i_1} + \cdots + \\ + \lambda^l \frac{p!}{l!(p-l)!}\Omega_{i_1\cdots i_l[a_{l+1}\cdots a_p}\partial_{a_1}\phi^{i_1}\cdots\partial_{a_l]}\phi^{i_l} + \cdots + \\ + \lambda^p \Omega_{i_1\cdots i_p}\partial_{[a_1}\phi^{i_1}\cdots\partial_{a_p]}\phi^{i_p} . \tag{2.4}$$

We are going to use this equation for $\Omega_p \equiv \frac{(i\lambda i_\phi i_\phi)^k}{k!}C_{p+2k}$. Combining eqs. (2.2) and (2.4), one gets the C_{p+2k} coupling of N D_{p-1}-branes (for $F_{ab}=0$) as:

$$\sum_{l,n} \frac{\mu_{p-1}\lambda^{k+n+l} i^k p!}{k!n!l!(p-l)!}\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l[a_{l+1}\cdots a_p} \times \\ \times \operatorname{Str}\left(\partial_{a_1}\phi^{j_1}\cdots\partial_{a_l]}\phi^{j_l}\phi^{i_1}\cdots\phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots\right), \tag{2.5}$$

where $0 \le l \le p$.

Notice that the $\operatorname{Str}(\cdots)$ expression involves symmetrizing over all the $\partial_{a_s}\phi^{j_s}$, for $s = 1, 2, \ldots, l$, also over all the ϕ^{i_q}, for $q = 1, 2, \ldots, n$, and all the pairs $\phi^{i'_{2j}}\phi^{i'_{2j-1}}$, for $j = 1, 2, \ldots, k$. We can rewrite this term as $\mu_{p-1}\sum_{l,n}\lambda^{k+n+l}b^n_l$, where

$$b^n_l = \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}\right)C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l[a_{l+1}\cdots a_p} \times \\ \times \operatorname{Str}\left(\partial_{a_1}\phi^{j_1}\cdots\partial_{a_l]}\phi^{j_l}\phi^{i_1}\cdots\phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots\right).$$

In the previous equation we antisymmetrized over all the a's, and this will be implicit in the rest of this note.

In order to show that the coupling is invariant, up to a total derivative, under gauge transformations $C \to C + d\Lambda$, we will try to write $\sum_{l,n} b^n_l$ as a sum of total derivatives and gauge invariant terms that depend on the field strength of the RR field. Integrating $b^n_{l>0}$ by parts with respect to σ^{a_l}, and dropping the resulting total derivatives and field strength terms, we can express $b^n_{l>0}$ as a sum of two types of terms. (we will keep track of the field strength terms and will present them later.) The first type of term for b^n_l will cancel against the second type of term in the expansion for b^{n+1}_{l-1}. In this way all the terms cancel, except for the first term in $b^n_{l=1}$ and $b^n_{l=0}$. (The second term of b^n_l will turn out to be 0 for $l = l_{\max} = p$, or for $n = n_{\min} = 0$.)

When integrating b^n_l by parts with respect to σ^{a_l}, we will get terms in which ∂_{a_l} acts either outside the trace on C^0, or inside on ϕ's. For the part inside the trace, for simplicity of notation, we will only write down the ϕ terms that have changed after integration by parts. Note that due to the antisymmetry in the a's, $\partial_{a_l}\partial_{a_s}\phi^j \to 0$. Let's denote by U^n_l the factor outside the trace,

$$U^n_l = \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}\right)C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l[a_{l+1}\cdots a_p} .$$

With these conventions, dropping the total derivative part,

$$\begin{aligned}
b_l^n &= (U_l^n)\,\mathrm{Str}\left(\cdots\partial_{a_l}\phi^{j_l}\cdots\right) = (-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\right) - \\
&\quad - k\left[(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots\right) + \right.\\
&\qquad\left. + (U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\phi^{i'_{2k}}\partial_{a_l}\phi^{i'_{2k-1}}\cdots\right)\right] - \\
&\quad - n(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l}\phi^{i_n}\cdots\right). \qquad (2.6)
\end{aligned}$$

The factor of k comes from the k pairs of $\phi^{j'}\phi^{j'-1}$ of the inner product and n from the n ϕ's of the Taylor series expansion of the RR form. Let:

$$\begin{aligned}
A_1 &= (U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots\right), \\
A_2 &= (U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\phi^{i'_{2k}}\partial_{a_l}\phi^{i'_{2k-1}}\cdots\right), \\
D_l^n &= (U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l}\phi^{i_n}\cdots\right), \\
b_l^n &= (-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\right) - k(A_1+A_2) - nD_l^n. \qquad (2.7)
\end{aligned}$$

Writing,

$$\partial_{x^{i_n}}C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l a_{l+1}\cdots a_p} = (p+2k+1)\partial_{[x^{i_n}}C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l a_{l+1}\cdots a_p]} + \partial_{x^{i'_1}}C^0_{i_n i'_2\cdots a_p} + [(p+2k-1)$$

more terms obtained interchanging i_n with all the other indices],
we can rewrite D_l^n as:

$$\begin{aligned}
D_l^n &= (\text{gauge invariant term}) + D_l^n|_{i_n\leftrightarrow i'_1} + \cdots + D_l^n|_{i_n\leftrightarrow i'_{2k}} + D_l^n|_{i_n\leftrightarrow j_1} + \cdots + \\
&\quad + D_l^n|_{i_n\leftrightarrow j_l} + D_l^n|_{i_n\leftrightarrow a_{l+1}} + \cdots + D_l^n|_{i_n\leftrightarrow a_p} \qquad (2.8)
\end{aligned}$$

(notation: $D_l^n|_{i_n\leftrightarrow i_s}$ means that in the term outside the symmetrized trace, i.e. U_l^n, we interchange i_n with i_s, and if these are dummy indices, this is equivalent to keeping the outside term U_l^n the same while interchanging i_n with i_s inside the trace.) Note that every time we interchange two indices from the set of indices i' and j inside the trace we get a minus sign since these indices are contracted with indices of the RR form in the term ouside the trace.

Since,

$$D_l^n = \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}\right)C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l[a_{l+1}\cdots a_p}\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l]}\phi^{i_n}\cdots\right)$$

we have the following relations:

$$D_l^n|_{i_n\leftrightarrow j_l} = b_l^n\,,$$

the original term.

$$\begin{aligned}
D_l^n|_{i_n\leftrightarrow j_s} &= (U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_l}\phi^{j_s}\cdots\partial_{a_s}\phi^{i_n}\cdots\right) \\
&= -(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\partial_{a_s}\phi^{j_s}\cdots\partial_{a_l}\phi^{i_n}\cdots\right) = -D_l^n\,,
\end{aligned}$$

for any $s = 1, \ldots, l-1$, from the antisymmetry in the a's.

$$D_l^n|_{i_n \leftrightarrow i'_{2k}} + D_l^n|_{i_n \leftrightarrow i'_{2k-1}} = D_l^n|_{i_n \leftrightarrow i'_{2j}} + D_l^n|_{i_n \leftrightarrow i'_{2j-1}}$$

Using the above relations, eq. (2.8) becomes

$$\begin{aligned} D_l^n &= (\text{gauge invariant term}) + k(D_l^n|_{i_n \leftrightarrow i'_{2k}} + D_l^n|_{i_n \leftrightarrow i'_{2k-1}}) - (l-1)D_l^n + b_l^n + \\ &\quad + D_l^n|_{i_n \leftrightarrow a_{l+1}} + \cdots + D_l^n|_{i_n \leftrightarrow a_p} \,. \end{aligned} \tag{2.9}$$

Dropping the gauge invariant term,

$$l(D_l^n) = k(B_1 + B_2) + b_l^n + D_l^n|_{i_n \leftrightarrow a_{l+1}} + \cdots + D_l^n|_{i_n \leftrightarrow a_p} \,, \tag{2.10}$$

where we have defined:

$$\begin{aligned} B_1 &= D_l^n|_{i_n \leftrightarrow i'_{2k}} = (U_l^n)\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots \underbrace{\phi^{i_n}\phi^{i'_{2k-1}}} \cdots \partial_{a_l}\phi^{i'_{2k}} \cdots\right), \\ B_2 &= D_l^n|_{i_n \leftrightarrow i'_{2k-1}} = (U_l^n)\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots \underbrace{\phi^{i'_{2k}}\phi^{i_n}} \cdots \partial_{a_l}\phi^{i'_{2k-1}} \cdots\right). \end{aligned}$$

The notation $\underbrace{\phi^i\phi^j}$ means that ϕ^i, ϕ^j show up together, as one entry, in the symmetrized trace prescription. In this way, the prescription, after interchanging some of the indices inside the trace, is consistent with the initial one.

Using the last equation to replace the D_l^n term in eq. (2.7), we find:

$$\begin{aligned} b_l^n &= (-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots\right) - k(A_1 + A_2) - \\ &\quad - \frac{n}{l}[k(B_1 + B_2) + b_l^n + D_l^n|_{i_n \leftrightarrow a_{l+1}} + \cdots + D_l^n|_{i_n \leftrightarrow a_p}] \,, \end{aligned} \tag{2.11}$$

$$\begin{aligned} b_l^n(n+l) &= l(-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots\right) - k(l(A_1 + A_2) + n(B_1 + B_2)) - \\ &\quad - n(D_l^n|_{i_n \leftrightarrow a_{l+1}} + \cdots + D_l^n|_{i_n \leftrightarrow a_p}) \,. \end{aligned} \tag{2.12}$$

Note that $(-\partial_{a_l})(U_l^n)\,\mathrm{Str}(\cdots \phi^{j_l} \cdots) = (-\partial_{a_s})(U_l^n)\,\mathrm{Str}(\cdots \phi^{j_s} \cdots)$, for any $s = 1, 2, \ldots, l-1$; we get a minus sign from $a_l \leftrightarrow a_s$, and another minus sign from $j_s \leftrightarrow j_l$.

Let's evaluate, $l(A_1 + A_2) + n(B_1 + B_2) = (lA_1 + nB_1) + (lA_2 + nB_2)$,

$$\begin{aligned} lA_1 + nB_1 &= lU_l^n\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots \underbrace{\partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}}} \cdots\right) + \\ &\qquad + nU_l^n\,\mathrm{Str}\left(\cdots \phi^{j_l} \cdots \underbrace{\phi^{i_n}\phi^{i'_{2k-1}}} \cdots \partial_{a_l}\phi^{i'_{2k}} \cdots\right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} lA_1 + nB_1 &= lU_l^n\,\mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\partial_{a_l}\phi^{j_l}\phi^{i'_{2k}}} \cdots\right) + \\ &\qquad + nU_l^n\,\mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\phi^{i_n}\phi^{i'_{2k}}} \cdots \partial_{a_l}\phi^{j_l} \cdots\right). \end{aligned} \tag{2.14}$$

All the Str$(\cdots)$ terms are multiplied by $U_l^n = \frac{i^k p!}{k!n!l!(p-l)!}(\partial_{x^{i_1}} \cdots \partial_{x^{i_n}}) C^0_{i'_1 \cdots i'_{2k} j_1 \cdots j_l [a_{l+1} \cdots a_p}$. Using the antisymmetry in the a's, and the symmetries in the dummy indices of the factor outside the trace, we have,

$$\begin{aligned} lA_1 + nB_1 = &(U_l^n) \sum_{q=1}^{l} \mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\partial_{a_q}\phi^{j_q}\phi^{i'_{2k}}} \cdots\right) + \\ &+ (U_l^n) \sum_{s=1}^{n} \mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\phi^{i_s}\phi^{i'_{2k}}} \cdots\right). \end{aligned} \tag{2.15}$$

Let's denote by Str$(\cdots \underbrace{\phi^{i'_{2k-1}}} \cdots \underbrace{\phi^{i'_{2k}}} \cdots)$, the expression in which $\phi^{i'_{2k-1}}$ and $\phi^{i'_{2k}}$ are distinct entries in the symmetrized trace prescription, without any constraint on the "left neighbour" of $\phi^{i'_{2k}}$. However, the "left neighbour" can only be one of the following: $\partial_{a_q}\phi^{j_q}|_{q=1,\cdots,l}$, or $\phi^{i_s}|_{s=1,\cdots,n}$, or $(\phi^{i'_{2j}}\phi^{i'_{2j-1}})|_{j=1,\cdots k-1}$, or $\phi^{i'_{2k-1}}$.

$$\begin{aligned} lA_1 + nB_1 = &(n+l+k)U_l^n \,\mathrm{Str}\left(\cdots \underbrace{\phi^{i'_{2k-1}}} \cdots \underbrace{\phi^{i'_{2k}}} \cdots\right) - \\ &- \sum_j U_l^n \,\mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k}}} \cdots\right) - \\ &- U_l^n \,\mathrm{Str}\left(\cdots \underbrace{\phi^{i'_{2k-1}}\phi^{i'_{2k}}} \cdots\right). \end{aligned} \tag{2.16}$$

One can notice that the last term is b_l^n, while the first term is 0, since U_l^n is antisymmetric in i'_{2k-1} and i'_{2k}.

$$lA_1 + nB_1 = b_l^n - (k-1)U_l^n \,\mathrm{Str}\left(\cdots \phi^{i'_{2k-1}} \cdots \underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k}}} \cdots\right). \tag{2.17}$$

We can ilustrate the type of identity that we used with a concrete example:

$$\mathrm{Str}\left(XY\,\underbrace{ZT}\right) = 3\,\mathrm{Str}\,(XYZT) - \mathrm{Str}\left(XZ\,\underbrace{YT}\right) - \mathrm{Str}\left(YZ\,\underbrace{XT}\right),$$

where X, Y, Z, T are some $N \times N$ matrices.
Similarly for,

$$\begin{aligned} nB_2 + lA_2 = &nU_l^n \,\mathrm{Str}\left(\cdots \phi^{i'_{2k}} \cdots \underbrace{\phi^{i'_{2k-1}}\phi^{i_n}} \cdots \partial_{a_l}\phi^{j_l} \cdots\right) + \\ &+ lU_l^n \,\mathrm{Str}\left(\cdots \phi^{i'_{2k}} \cdots \underbrace{\phi^{i'_{2k-1}}\partial_{a_l}\phi^{j_l}} \cdots\right), \end{aligned} \tag{2.18}$$

$$\begin{aligned} nB_2 + lA_2 = &(n+l+k)U_l^n \,\mathrm{Str}\left(\cdots \underbrace{\phi^{i'_{2k}}} \cdots \underbrace{\phi^{i'_{2k-1}}} \cdots\right) - \\ &- (k-1)U_l^n \,\mathrm{Str}\left(\cdots \underbrace{\phi^{i'_{2k-1}}\phi^{i'_{2j}}\phi^{i'_{2j-1}}} \cdots\right) - \\ &- U_l^n \,\mathrm{Str}\left(\cdots \underbrace{\phi^{i'_{2k-1}}\phi^{i'_{2k}}} \cdots\right), \end{aligned} \tag{2.19}$$

$$nB_2 + lA_2 = b_l^n - (k-1)U_l^n \,\mathrm{Str}\left(\cdots \phi^{i'_{2k}} \cdots \underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k-1}}} \cdots\right). \tag{2.20}$$

From eqs. (2.17) and (2.20), we get:

$$l(A_1+A_2)+n(B_1+B_2)=2b_l^n\,. \tag{2.21}$$

Then eq. (2.12) gives,

$$\begin{aligned} b_l^n &= \frac{l}{n+l+2k}(-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\right)-\frac{n}{n+l+2k}\left[D_l^n|_{i_n\leftrightarrow a_{l+1}}+\cdots+\right.\\ &\qquad\qquad\left.+D_l^n|_{i_n\leftrightarrow a_p}\right]. \end{aligned} \tag{2.22}$$

Let's remind ourselves what these terms really are,

$$\begin{aligned} (-\partial_{a_l})(U_l^n)\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\right) &= \frac{-i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}\partial_{a_l}\right)C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l a_{l+1}\cdots a_p}\times\\ &\quad\times\,\mathrm{Str}\left(\partial_{a_1}\phi^{j_1}\cdots\partial_{a_{l-1}}\phi^{j_{l-1}}\underline{\phi}^{j_l}\phi^{i_1}\cdots\phi^{i_n}\times\right.\\ &\qquad\left.\times\,\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots}\right), \end{aligned} \tag{2.23}$$

$$\begin{aligned} D_l^n|_{i_n\leftrightarrow a_{l+1}} &= \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\cdots\partial_{x^{i_{n-1}}}\partial_{a_{l+1}}\right)C^0_{i'_1\cdots i'_{2k}j_1\cdots j_l i_n a_{l+2}\cdots a_p}\times\\ &\quad\times\,\mathrm{Str}\left(\partial_{a_1}\phi^{j_1}\cdots\partial_{a_{l-1}}\phi^{j_{l-1}}\underline{\phi}^{j_l}\phi^{i_1}\cdots\phi^{i_{n-1}}\partial_{a_l}\phi^{i_n}\times\right.\\ &\qquad\left.\times\,\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots}\right). \end{aligned} \tag{2.24}$$

If we use the same expansion for b_{l+1}^{n-1}, we get,

$$\begin{aligned} b_{l+1}^{n-1} &= \frac{l+1}{n+l+2k}(-\partial_{a_{l+1}})(U_{l+1}^{n-1})\,\mathrm{Str}\left(\cdots\phi^{i_{n+l+1}}\cdots\right)-\\ &\quad-\frac{n-1}{n+l+2k}\left[D_{l+1}^{n-1}|_{i_n\leftrightarrow a_{l+2}}+\cdots+D_{l+1}^{n-1}|_{i_n\leftrightarrow a_p}\right]. \end{aligned} \tag{2.25}$$

We can see that the second term in the expression for b_l^n is the same as the first in the expression for b_{l+1}^{n-1}. All the $D_l^n|_{(\dots)}$ terms from (2.22) are equal to each other, due to the antisymmetry in the a's, and as it turns out, they come with the right sign to cancel the first term from (2.25). Let's check the numerical coefficients,

$$\text{second term of } b_l^n:\ \frac{p!}{k!n!l!(p-l)!}\frac{n}{n+l+2k}(p-l)\,.$$

Note that this is 0, for $l=l_{\max}=p$, or for $n=n_{\min}=0$.

$$\text{first term of } b_{l+1}^{n-1}:\ \frac{p!}{k!(n-1)!(l+1)!(p-l-1)!}\frac{l+1}{n+l+2k}\,.$$

The numerical factors are the same so all the terms cancel against each other, except for the b_0^n, and the first term of b_1^n given by eq. (2.22). The $l=0$ case has to be analysed separately, since we cannot integrate by parts in this case.

$$b_0^n = \frac{1}{k!n!}(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}})C^0_{i'_1\cdots i'_{2k}a_1\cdots a_p}\,\mathrm{Str}\left(\phi^{i_1}\cdots\phi^{i_n}\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right). \tag{2.26}$$

As in eq. (2.16), we can write:

$$\begin{aligned}0 &= (n+k)U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}}\cdots\underbrace{\phi^{i'_{2k-1}}}\cdots\right)\\ &= nU_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\cdots\right)+(k-1)U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k-1}}}\cdots\right)+\\ &\quad+U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right).\end{aligned} \tag{2.27}$$

Similarly,

$$\begin{aligned}0 &= nU_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i_n}}\cdots\right)+(k-1)U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2j}}\phi^{i'_{2j-1}}}\cdots\right)+\\ &\quad+U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right).\end{aligned} \tag{2.28}$$

From (2.27) and (2.28) we have:

$$\begin{aligned}2U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right) &= -nU_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\cdots\right)-\\ &\quad-nU_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i_n}}\cdots\right)\\ &= -n(U_0^n|_{i_n\leftrightarrow i'_{2k}}+U_0^n|_{i_n\leftrightarrow i'_{2k-1}})\times\\ &\quad\times\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right).\end{aligned} \tag{2.29}$$

Repeating this k times, we end up with:

$$\begin{aligned}U_0^n\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right) &= \frac{n}{n+2k}(U_0^n-U_0^n|_{i_n\leftrightarrow i'_1}-\cdots-U_0^n|_{i_n\leftrightarrow i'_{2k}})\times\\ &\quad\times\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right).\end{aligned} \tag{2.30}$$

From (2.22) we find that the first term of b_1^{n-1} is (rename $j_1\to i_n$):

$$\begin{aligned}&-\frac{i^k p}{k!(n-1)!(n+2k)}\partial_{x^{i_1}}\cdots\partial_{x^{i_{n-1}}}\partial_{a_1}C^0_{i'_1\cdots i'_{2k}i_n a_2\cdots a_p}\times\\ &\quad\times\,\mathrm{Str}\left(\phi^{i_1}\cdots\phi^{i_n}\cdots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right).\end{aligned} \tag{2.31}$$

Now we can sum the b_0^n term and the first term of b_1^{n-1} to get a gauge invariant term equal to:

$$\frac{i^k}{k!(n-1)!(n+2k)}\partial_{x^{i_1}}\cdots\partial_{x^{i_{n-1}}}F^{0,(2k+p+1)}_{i_1 i'_1\cdots i'_{2k}a_1\cdots a_p}\,\mathrm{Str}\left(\phi^{i_1}\cdots\phi^{i_n}\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\cdots\right),$$

where $F^{0,(2k+p+1)} = dC^{0,(2k+p)}$. The monopole coupling doesn't show up in the previous expression, since in deriving eq. (2.30) we assumed $k > 0$. Keeping track of the gauge invariant terms dropped in eq. (2.10), we can express the total coupling, for $F = 0$, as:

$$\begin{aligned}&\mu_{p-1}\sum_{l,n>0}\frac{\lambda^{(k+n+l)}i^k p!}{k!(n-1)!l!(p-l)!(2k+n+l)}\partial_{x^{i_1}}\cdots\partial_{x^{i_{n-1}}}F^{0,(2k+p+1)}_{i_1 i'_1\cdots i'_{2k}j_1\cdots j_l a_{l+1}\cdots a_p}\times\\&\quad\times\,\mathrm{Str}\left(\partial_{a_1}\phi^{j_1}\cdots\partial_{a_l}\phi^{j_l}\phi^{i_1}\cdots\phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\cdots\right).\end{aligned}\tag{2.32}$$

For $k = 0$ we need to add the usual monopole coupling given by $\mu_{p-1}C^0_{a_1\cdots a_p}$ from (2.26).

3. Gauge invariance for $F \neq 0$

For non-zero F along the brane, the pull-back is defined using covariant derivatives. There are a few useful relations involving covariant derivatives that allow us to use the previous proof in the case when F is non zero. If Y, Y_1, Y_2 are $N\times N$ matrices transforming in the adjoint representation of the gauge group ($D_aY = \partial_aY + i[A_a, Y]$), and f is a scalar function, then:

(a) $\mathrm{Tr}[D_a(Y_1Y_2)] = \mathrm{Tr}[D_a(Y_1)Y_2] + \mathrm{Tr}[Y_1(D_aY_2)]$,

(b) $D_a(fY) = (\partial_af)Y + fD_aY$,

(c) $[D_1, D_2]Y = i[F_{12}, Y]$, where $F_{12} = \partial_1A_2 - \partial_2A_1 + i[A_1, A_2]$,

(d) $D_{[a}F_{bc]} = 0$, by the Bianchi identity

In this case, the equivalent of eq. (2.5), which gives the coupling between N D$_{p-1}$-branes and C_{p+2k} is:

$$\begin{aligned}&\sum_{l,n,r}\frac{\mu_{p-1}\lambda^{k+n+l+2r}i^{k+r}p!}{2^r r!(k+r)!(n)!l!(p-l-2r)!}\partial_{x^{i_1}}\cdots\partial_{x^{i_n}}C^0_{i'_1\cdots i'_{2(k+r)}j_1\cdots j_l[a_{l+1}\cdots a_{p-2r}}\times\\&\quad\times\,\mathrm{Str}\left(D_{a_1}\phi^{j_1}\cdots D_{a_l}\phi^{j_l}\phi^{i_1}\cdots\phi^{i_n}\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\cdots\underbrace{\phi^{i'_2}\phi^{i'_1}}\cdots F_{a_pa_{p-1}]}\right),\end{aligned}\tag{3.1}$$

where $0 \le l \le p-2r$, and r is the number of F's appearing inside the Str part. As in the proof for $F = 0$, we will write the sum in (3.1) as $\mu_{p-1}\sum_{l,n,r}\lambda^{k+n+l+2r}b^n_{l,r}$, and denote by $U^n_{l,r}$ the term outside the trace corresponding to $b^n_{l,r}$. When integrating $b^n_{l,r}$ by parts, now we will have extra terms containing $\mathrm{Str}(\cdots D_{[a_1}D_{a_2]}\phi\cdots)$. Since $D_{[a_1}D_{a_2]}\phi = \frac{1}{2}[D_{a_1},D_{a_2}]\phi = \frac{i}{2}[F_{a_1a_2},\phi]$ these extra terms will cancel against other terms in the expansion for $b^n_{l-2,r+1}$. Given these facts, the right-hand side of eq. (2.6) has an aditional term equal to:

$$(l-1)(-U^n_{l,r})\,\mathrm{Str}\left(\phi^{j_1}\cdots D_{a_1}D_{a_l}\phi^{j_l}\cdots\right),$$

while (2.17) changes to:

$$\begin{aligned} lA_1+nB_1 = b^n_{l,r} &- (k+r-1)U^n_{l,r}\,\mathrm{Str}\left(\cdots\phi^{i'_{2(k+r)-1}}\cdots\underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2(k+r)}}}\cdots\right) - \\ &- rU^n_{l,r}\,\mathrm{Str}\left(\cdots\underbrace{F\phi^{i'_{2(k+r)}}}\cdots\right). \end{aligned}$$

Now, we can see that the generalizations of eqs. (2.21) and (2.22) are:

$$l(A_1+A_2)+n(B_1+B_2) = 2b^n_{l,r} - rU^n_{l,r}\,\mathrm{Str}\left(\cdots\underbrace{[F,\phi^{i'_{2(k+r)}}]}\cdots\right) \tag{3.2}$$

$$\begin{aligned} b^n_{l,r} = &\frac{l}{n+l+2(k+r)}(-\partial_{a_l})(U^n_{l,r})\,\mathrm{Str}\left(\cdots\phi^{j_l}\cdots\right) - \\ &-\frac{l(l-1)}{n+l+2(k+r)}(U^n_{l,r})\,\mathrm{Str}\left(\phi^{j_1}\cdots D_{a_1}D_{a_l}\phi^{j_l}\cdots\right) - \\ &-\frac{n}{n+l+2(k+r)}\left[D^n_{l,r}|i_n\leftrightarrow a_{l+1}+\cdots+\right.\\ &\qquad\qquad\left.+D^n_{l,r}|i_n\leftrightarrow a_{p-2r}\right]+\\ &+\frac{r(k+r)}{n+l+2(k+r)}(U^n_{l,r})\,\mathrm{Str}\left(\cdots\underbrace{[F,\phi^{i'_{2(k+r)}}]}\cdots\right). \end{aligned} \tag{3.3}$$

The second term in the expansion for $b^n_{l,r}$ has the same structure as the fourth term in the expansion for $b^n_{l-2,r+1}$. While it is easy to see that these extra terms have the required form to produce the (partial) cancelation between $b^n_{l,r}$, and $b^n_{l-2,r+1}$ we have to make sure the numerical pre-factors are equal:

$$\begin{aligned} &\text{from } b^n_{l,r}: \ \frac{i^{k+r}p!}{2^r(k+r)!r!(n)!l!(p-l-2r)!}l(l-1)\frac{i}{2}, \\ &\text{from } b^n_{l-2,r+1}: \ \frac{i^{k+r+1}p!}{2^{r+1}(k+r+1)!(r+1)!n!(l-2)!(p-l-2r)!}(r+1)(k+r+1). \end{aligned}$$

Since the numerical factors are the same, when we are summing the $b^n_{l,r}$'s over r, all the extra terms that we get in the case of a non-zero F will cancel against each other, except for the second term of $b^n_{l=2,r}$. At the limits, when $r=r_{\min}$ the fourth term in the expression for $b^n_{l,r}$ is 0, since $r_{\min}(r_{\min}+k)=0$. If $k>0$, $r_{\min}=0$, otherwise

$r_{\min} = -k$. When $r = r_{\max}$, $l < 2$ so the second term in the expansion for $b^n_{l,r}$ is 0. After summing over l and r we are left with:

$$\sum_r (b^n_{l=0,r}) + \sum_r \frac{1}{n+2(k+r)}(-\partial_{a_1})(U^{n-1}_{l=1,r}) \,\mathrm{Str}\left(\cdots\phi^{j_1}\cdots\right) + \\ + \sum_{r>r_{\min}} \frac{-i}{n+2(k+r)} U^n_{2,r-1}\,\mathrm{Str}\left(\cdots\phi^{j_1}\cdots\underbrace{[F_{a_1 a_l},\phi^{j_l}]}\cdots\right). \qquad (3.4)$$

For $b_{l=0,r}$ we are using a transformation as in (2.30):

$$U^n_{0,r}\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\cdots\right) = \\ = \frac{n}{n+2(k+r)}\left(U^n_{0,r} - U^n_{0,r}|_{i_n\leftrightarrow i'_1} - \cdots - U^n_{0,r}|_{i_n\leftrightarrow i'_{2(k+r)}}\right)\times \\ \times\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\cdots\right) - \frac{(k+r)(r)}{n+2(k+r)}U^n_{0,r}\times \\ \times\,\mathrm{Str}\left(\cdots\underbrace{[F,\phi^{i'_{2(k+r)-1}}]}\cdots\right). \qquad (3.5)$$

Using (3.5) to replace $b^n_{l=0,r}$ in (3.4), we get:

$$\sum_r \frac{n}{n+2(k+r)}(U^n_{0,r} - U^n_{0,r}|_{i_n\leftrightarrow i'_1} - \cdots - U^n_{0,r}|_{i_n\leftrightarrow i'_{2(k+r)}})\times \\ \times\,\mathrm{Str}\left(\cdots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\cdots\right) + \sum_r \frac{1}{n+2(k+r)}(-\partial_{a_1})(U^{n-1}_{l=1,r})\times \\ \times\,\mathrm{Str}\left(\cdots\phi^{j_1}\cdots\right).$$

The above expression is gauge invariant since we can write it as a field strength term noticing that after renaming $j_1 \to i_n$, the Str parts are identical and:

$$(\partial_{a_1})(U^{n-1}_{l=1,r}) = n(p-2r)U^n_{0,r}|_{i_n\leftrightarrow a_1}\,.$$

Taking into account the corresponding gauge invariant terms dropped in eq. (2.10) the total coupling between N D$_{p-1}$-branes and a C_{p+2k} potential can be expressed in a gauge invariant way as:

$$\mu_{p-1}\sum_{r,l}\frac{\lambda^{k+1+2r+l}i^{k+r}p!}{2^r r!(k+r)!l!(p-2r-l)!}\times \\ \times\,\mathrm{Str}\left(\overline{F}^{(2k+p+1)}_{r,l}(\phi)_{i_1 i'_1 i'_2\cdots i'_{2(k+r)-1}i'_{2(k+r)}j_1\cdots j_l a_{l+1}\cdots a_{p-2r}}\phi^{i_1}D_{a_1}\phi^{j_1}\cdots D_{a_l}\phi^{j_l}\times\right. \\ \left.\times\,\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\cdots\underbrace{\phi^{i'_2}\phi^{i'_1}}F_{a_{p-2r+1}a_{p-2r+2}}\cdots F_{a_{p-1}a_p}\right). \qquad (3.6)$$

where we defined,

$$\overline{F}^{(2k+p+1)}_{r,l}(\phi) = \sum_{n\geq 0} \frac{\lambda^n}{(n)!(n+l+2k+2r+1)} \phi^{i_1} \cdots \phi^{i_n} \partial_{x^{i_1}} \cdots \partial_{x^{i_n}} F^{0,(2k+p+1)}(\sigma, x^i)|_{x^i=0} \,. \tag{3.7}$$

Since eq. (3.5) was derived assuming $k > 0$, for $k \leq 0$ there is an additional monopole coupling term given by:

$$\mu_{p-1} \frac{\lambda^{|k|} p!}{2^{|k|}(|k|)!(p-2|k|)!} C^0_{[a_1 \cdots a_{p-2|k|}} F \cdots F_{a_{p-1} a_p]} \,. \tag{3.8}$$

4. Discussion

We obtained a manifestly gauge invariant expression for the Chern-Simons coupling between N D$_{p-1}$-branes and a RR potential, C_{p+2k}. In the presence of a 2-form B-field, the gauge transformations of the RR fields, become:

$$\sum_n C^{(n)} e^B \longrightarrow \sum_n C^{(n)} e^B + d \sum_p \Lambda^{(p)} \,. \tag{4.1}$$

The presence of the B-field does not affect the generality of the previous proof since, from the point of view of the gauge transformations, we can absorb B into the definition of the RR fields. However, the proof in this note applies only for finite N. For $N \to \infty$ we can no longer use the property of cyclicity of the trace, and we expect monopole couplings even to higher rank RR fields. As in matrix model, one can construct a higher dimensional brane out of an infinite number of lower dimensional ones, hence in (1.1) we should have source terms for higher dimensional D-brane charges.

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After this work was completed, the paper [8] appeared. It might shed new insight on the issues we studied in this note.

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